Fourier Analysis

Feb. 20, 2024

Review:

A convergence Thm: Let f be integrable on the circle.

Suppose f is differentiable at x_0. Then

$$S_N f(x_0) \to f(x_0) \quad \text{as } N \to \infty.$$

The above result is based on the Riemann-Lebesgue lemma.

Remark: Using the same proof, we can show that the above result holds under the weaker assumption that

f is Lipschitz continuous at x_0, this is,

$$\exists\ M > 0 \text{ such that}$$

$$(*) \qquad |f(x) - f(x_0)| \leq M \cdot |x - x_0| \quad \text{for all } x \in [-\pi, \pi].$$

Corollary 1. Let $f, g \in \mathcal{R}$. Suppose that f and g coincide on an open interval I which contains x_0. Then

$$S_N f(x_0) - S_N g(x_0) \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

Pf. Apply the convergence Thm to $f-g$ at x_0. $\square$.

FACT: The above result implies that the convergence of $S_N f$ at x_0 only depends on the behavior of f on a neighborhood of x_0.

This fact is called "Riemann's localization principle".

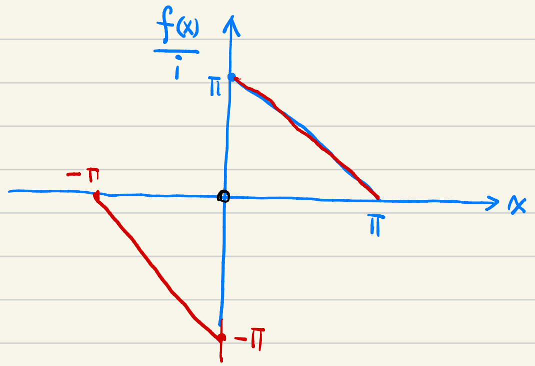
§ 3.2

A continuous function on the circle whose Fourier series diverges at some point.

Below we construct a cts function g on the circle s.t. $S_N g(0) \not\to g(0)$.

Let us start from a special function

$$f(x) = \begin{cases} i(\pi - x) & \text{if } 0 \leq x \leq \pi \\ i(-\pi - x) & \text{if } -\pi \leq x < 0 \end{cases}$$



A direct calculation shows that

$$f(x) \sim \sum_{n \neq 0} \frac{1}{n} e^{inx} \qquad \text{on } [-\pi, \pi]$$

For $N \in \mathbb{N}$, define

$$f_N(x) = \sum_{0 < |n| \le N} \frac{1}{n} e^{inx} \qquad (= S_N f(x))$$

$$\tilde{f}_N(x) = \sum_{n=-N}^{-1} \frac{1}{n} e^{inx}$$

Lem 2 : (1) $\exists$ a constant $M > 0$ such that

$$|f_N(x)| \le M \quad \text{for all } N \in \mathbb{N} \text{ and } x \in [-\pi, \pi]$$

(2) $$|\tilde{f}_N(0)| \ge \log N$$

Pf. We first prove (1).

Let us consider the Abel mean of f,

$$A_r f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x-y) f(y)\, dy,$$

where $P_r(x) := \dfrac{1 - r^2}{1 - 2r\cos x + r^2}$, $0 \le r < 1$.

Notice that for $0 \leq r < 1$,

$$|A_r f(x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x-y) \, |f(y)| \, dy$$

$$\leq \frac{\|f\|_\infty}{2\pi} \int_{-\pi}^{\pi} P_r(x-y) \, dy$$

$$= \|f\|_\infty \qquad \left(\|f\|_\infty := \sup_{y \in [-\pi, \pi]} |f(y)| \right)$$

However

$$|f_N(x) - A_r f(x)|$$

$$= \left| \sum_{0 < |n| \leq N} \frac{1}{n} e^{inx} - \sum_{|n| > 0} \frac{r^{|n|}}{n} e^{inx} \right|$$

$$= \left| \sum_{0 < |n| \leq N} \frac{1 - r^{|n|}}{n} e^{inx} - \sum_{|n| \geq N+1} \frac{r^{|n|}}{n} e^{inx} \right|$$

$$\leq \left| \sum_{0 < |n| \leq N} \frac{1 - r^{|n|}}{n} e^{inx} \right| + \left| \sum_{|n| \geq N+1} \frac{r^{|n|}}{n} e^{inx} \right|$$

$$\leq \sum_{0<|n|\leq N} \frac{1-r^{|n|}}{|n|} + \sum_{|n|\geq N+1} \frac{r^{|n|}}{|n|}$$

$$\leq 2\cdot \sum_{n=1}^{N} \frac{1-r^{n}}{n} + 2\cdot \sum_{n=N+1}^{\infty} \frac{r^{n}}{N}$$

Notice that $\frac{1-r^n}{n} = \frac{(1-r)(1+r+r^2+\cdots+r^{n-1})}{n}$

$$\leq 1-r.$$

Hence

$$|f_N(x) - A_r f(x)| \leq 2N(1-r) + 2\sum_{n=N+1}^{\infty} \frac{r^n}{N}$$

$$\leq 2N(1-r) + 2\cdot \frac{r^{N+1}}{N(1-r)}$$

$$\leq 2N(1-r) + \frac{2}{N(1-r)}$$

Let us take $\tilde{r} = 1-\frac{1}{N}$, then $N(1-\tilde{r}) = 1$

Then $|f_N(x) - A_{\tilde{r}} f(x)| \leq 4$

It follows that

$$|f_N(x)| \leqslant 4 + |A_{\tilde{r}} f(x)|$$
$$\leqslant 4 + \|f\|_\infty$$

This prove (1).

Next we prove (2), i.e. $|\hat{f}_N(0)| > \log N$

Notice that

$$\hat{f}_N(0) = \sum_{n=-N}^{-1} \frac{1}{n} e^{in\cdot 0} = -\left[1 + \frac{1}{2} + \cdots + \frac{1}{N}\right]$$

So

$$|\hat{f}_N(0)| = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$$

(using $\frac{1}{k} \geq \int_k^{k+1} \frac{1}{x}\,dx$ for all $k > 0$)

$$\geqslant \sum_{k=1}^{N} \int_k^{k+1} \frac{1}{x}\,dx$$

$$= \sum_{k=1}^{N} \left(\log(k+1) - \log k \right)$$

$$= \log(N+1) > \log N.$$

Next for $N \in \mathbb{N}$, define

$$P_N(x) = e^{2iNx} \cdot f_N(x)$$

$$= \sum_{\substack{n=-N \\ n \neq 0}}^{N} \frac{1}{n} e^{i(n+2N)x}$$

$$= \sum_{\substack{n=N \\ n \neq 2N}}^{3N} \frac{1}{n-2N} e^{inx}$$

$$\tilde{P}_N(x) = e^{2iNx} \cdot \tilde{f}_N(x) = \sum_{n=N}^{2N-1} \frac{1}{n-2N} e^{inx}$$

Clearly $|P_N(x)| = |f_N(x)| \leq M$,

$|\tilde{P}_N(x)| = |\tilde{f}_N(x)|$, $|\tilde{P}_N(0)| > \log N$.

Construct a sequence $(N_k)_{k=1}^{\infty}$ of positive integers and a sequence $(\alpha_k)_{k=1}^{\infty}$ of positive numbers such that

(1) $N_{k+1} > 3 N_k$

(2) $\sum_{k=1}^{\infty} \alpha_k < \infty$

(3) $\alpha_k \log N_k \to \infty$ as $k \to \infty$.

(For instance, we can take

$$N_k = 4^{4^k} \quad \text{and} \quad \alpha_k = 3^{-k} \;)$$

Finally set

$$g(x) = \sum_{k=1}^{\infty} \alpha_k \, P_{N_k}(x), \qquad x \in [-\pi, \pi].$$

Since $|P_{N_k}(x)| \leq M$ and $\sum_{k=1}^{\infty} \alpha_k M < \infty$,

by the Weirestrass M-Test, we have

$$\sum_{k=1}^{\ell} a_k P_{N_k}(x) \rightrightarrows g \quad \text{on the circle}$$
as $\ell \to \infty$.

and g is continuous on the circle.

Next we claim that

$$S_N g(0) \not\to g(0) \quad \text{as } N \to \infty \quad (**)$$

Lem 3.

$$\hat{g}(n) = \begin{cases} \dfrac{a_k}{n - 2N_k} & \text{if } n \in [N_k, 3N_k] \text{ and } n \neq 2N_k \\ 0 & \text{otherwise.} \end{cases}$$

Pf. Recall that for given N,

$$P_N(x) = \sum_{\substack{N \le n \le 3N \\ n \neq 2N}} \frac{1}{n-2N} e^{inx}$$

Hence $$\hat{P}_N(n) = \begin{cases} \dfrac{1}{n-2N} & \text{if } n \in [N, 3N],\ n \neq 2N \\ 0 & \text{otherwise} \end{cases}$$

Now fix $n \in \mathbb{Z}$. Let $\varepsilon > 0$. Then $\exists\, L \in \mathbb{N}$ such that

$$\left| g(x) - \sum_{j=1}^{\ell} \alpha_j P_{N_j}(x) \right| < \varepsilon \quad \text{if } \ell \geq L.$$

Observe that for $h \in \mathcal{R}$,

$$|\hat{h}(n)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x) e^{-inx}| dx$$
$$\leq \|h\|_\infty.$$

Hence

$$(***) \quad \left| \hat{g}(n) - \sum_{j=1}^{\ell} \alpha_j \widehat{P_{N_j}}(n) \right| < \varepsilon \quad \text{if } \ell \geq L$$

Now suppose that

$$n \notin \bigsqcup_{j=1}^{\infty} [N_j, 3N_j],$$

Then $\alpha_j \widehat{P_{N_j}}(n) = 0$ for all j so

$$\sum_{j=1}^{\ell} \alpha_j \widehat{P_{N_j}}(n) = 0 \text{ for all } \ell.$$

So by (***),

$$|\hat{g}(n)| \leq \varepsilon.$$

Since ε is arbitrarily given, $\hat{g}(n) = 0$.

Next assume $n \in [N_k, 3N_k]$ for some k.

Then for any $l \geq k$,

$$\sum_{j=1}^{l} \alpha_j \hat{P}_{N_j}(n) = \alpha_k \hat{P}_{N_k}(n)$$

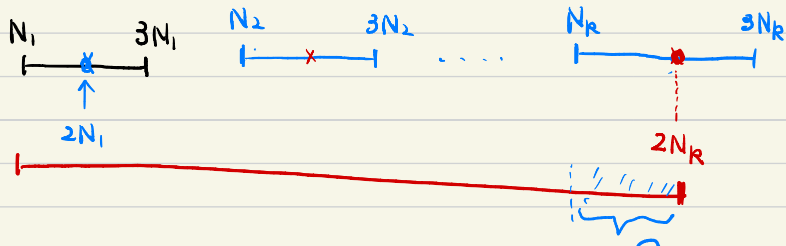
$$= \begin{cases} \alpha_k \cdot \dfrac{1}{n-2N_k} & \text{if } n \neq 2N_k \\ 0 & \text{if } n = 2N_k. \end{cases}$$

By (***), we see that

$$\hat{g}(n) = \alpha_k \hat{P}_{N_k}(n)$$

This proves Lem 3. $\boxed{}$

• $g(x) = \alpha_1 P_{N_1}(x) + \alpha_2 P_{N_2}(x) + \cdots + \alpha_k \cdot P_{N_k}(x) + \cdots$



Let us consider $S_{2N_k} g$

We notice that

$$S_{2N_k} g(x) = \alpha_1 P_{N_1}(x) + \cdots + \alpha_{k-1} P_{N_{k-1}}(x)$$
$$+ \alpha_k \cdot \widetilde{P_{N_k}}(x)$$

$$S_{2N_k} g(0) = \left(\alpha_1 P_{N_1}(0) + \cdots + \alpha_{k-1} P_{N_{k-1}}(0) \right)$$
$$+ \alpha_k \cdot \widetilde{P_{N_k}}(0)$$
$$= (\mathrm{I}) + (\mathrm{II})$$

$$|(\mathrm{I})| \leq \sum_{i=1}^{k-1} \alpha_i \cdot M < \sum_{i=1}^{\infty} \alpha_i$$

but $|(\mathrm{II})| \geq \alpha_k \log N_k \to +\infty$

Hence

$$\left| S_{2N_k} g(0) \right| \to +\infty \neq g(0).$$

$\blacksquare$